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Characterizing Jacobians via trisecants of the Kummer Variety

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Abstract

We prove Welters' trisecant conjecture: an indecomposable principally polarized abelian variety X is the Jacobian of a curve if and only if there exists a trisecant of its Kummer variety $K(X)$.

1 Introduction

Welters' remarkable trisecant conjecture formulated first in [1] was motivated by Gunning's celebrated theorem ([2]) and by another famous conjecture: the Jacobians of curves are exactly the indecomposable principally polarized abelian varieties whose theta-functions provide explicit solutions of the so-called KP equation. The latter was proposed earlier by Novikov and was unsettled at the time of the Welters' work. It was proved later by T.Shiota [3] and until recently has remained the most effective solution of the classical Riemann-Schottky problem.

Let B be an indecomposable symmetric matrix with positive definite imaginary part. It defines an indecomposable principally polarized abelian variety $X = \mathbb{C}^g/\Lambda$, where the lattice Λ is generated by the basis vectors $e_m \in \mathbb{C}^g$ and the column-vectors B_m of B. The Riemann theta-function $\theta(z) = \theta(z|B)$ corresponding to B is given by the formula

$$\theta(z) = \sum_{m\in\mathbb{Z}^g} e^{2\pi i(z,m)+\pi i(Bm,m)}, \quad (z,m) = m_1z_1 + \cdots + m_gz_g\,. \tag{1.1}$$

The Kummer variety $K(X)$ is an image of the Kummer map

$$K : Z \in X \longmapsto \{\Theta[\varepsilon_1, 0](Z) : \cdots : \Theta[\varepsilon_{2^g}, 0](Z)\} \in \mathbb{CP}^{2^g-1} \tag{1.2}$$

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where $\Theta[\varepsilon, 0](z) = \theta[\varepsilon, 0](2z|2B)$ are level two theta-functions with half-integer characteristics ε.

A trisecant of the Kummer variety is a projective line which meets $K(X)$ at least at three points. Fay's well-known trisecant formula [4] implies that if B is a matrix of b-periods of normalized holomorphic differentials on a smooth genus g algebraic curve Γ, then a set of three arbitrary distinct points A_1, A_2, A_3 on Γ defines a *one-parametric family* of trisecants parameterized by a fourth point of the curve $A_4 \neq A_1, A_2, A_3$. In [2] Gunning proved under certain non-degeneracy assumptions that the existence of such a family of trisecants characterizes Jacobian varieties among indecomposable principally polarized abelian varieties.

Gunning's geometric characterization of the Jacobian locus was extended by Welters who proved that the Jacobian locus can be characterized by the existence of a formal one-parametric family of flexes of the Kummer varieties [1, 5]. A flex of the Kummer variety is a projective line which is tangent to $K(X)$ at some point up to order 2. It is a limiting case of trisecants when the three intersection points come together.

In [6] Arbarello and De Concini showed that the Welters' characterization is equivalent to an infinite system of partial differential equations representing the so-called KP hierarchy, and proved that only a finite number of these equations is sufficient. In fact, the KP theory and the earlier results of Burchnall, Chaundy and the author [7, 8, 9, 10] imply that the Jacobian locus is characterized by the first $N = g + 1$ equations of the KP hierarchy, only. Novikov's conjecture that just the first equation ($N = 1!$) of the hierarchy is sufficient for the characterization of the Jacobians is much stronger. It is equivalent to the statement that the Jacobians are characterized by the existence of length 3 formal jet of flexes.

In [1] Welters formulated the question: *if the Kummer-Wirtinger variety $K(X)$ has one trisecant, does it follow that X is a Jacobian ?* In fact, there are three particular cases of the Welters' conjecture, which are independent and have to be considered separately. They correspond to three possible configurations of the intersection points (a, b, c) of $K(X)$ and the trisecant:

(i) all three points coincide ($a = b = c$),

(ii) two of them coincide ($a = b \neq c$);

(iii) all three intersection points are distinct ($a \neq b \neq c \neq a$).

The affirmative answer to the the first particular case (i) of the Welters' question was obtained in the author's previous work [12]. (Under various additional assumptions in various forms it was proved earlier in [13, 14, 15]). The aim of this paper is to prove, using the approach proposed in [12], the two remaining cases of the trisecant conjecture. It seems that the approach is very robust and can be applied to the variety of Riemann-Schottky-type problems. For example, in [16] it was used for the characterization of principally polarized Prym varieties of branched covers.

Our first main result is the following statement.

Theorem 1.1 *An indecomposable, principally polarized abelian variety (X, θ) is the Jacobian of a smooth curve of genus g if and only if there exist non-zero g-dimensional vectors $U \neq A \,(\mathrm{mod}\, \Lambda)$, V, such that one of the following equivalent conditions holds:*

(*A*) *The differential-difference equation*

$$(\partial_t - T + u(x,t))\,\psi(x,t) = 0, \quad T = e^{\partial_x} \tag{1.3}$$

is satisfied for

$$u = (T-1)v(x,t), \quad v = -\partial_t \ln\theta(xU + tV + Z) \tag{1.4}$$

and

$$\psi = \frac{\theta(A + xU + tV + Z)}{\theta(xU + tV + Z)}\, e^{xp+tE}, \tag{1.5}$$

where p, E *are constants and* Z *is arbitrary.*

(*B*) *The equations*

$$\partial_V\Theta[\varepsilon,0]\,((A-U)/2) - e^p\Theta[\varepsilon,0]\,((A+U)/2) + E\Theta[\varepsilon,0]\,((A-U)/2) = 0, \tag{1.6}$$

are satisfied for all $\varepsilon \in \frac{1}{2}Z_2^g$. *Here and below* ∂_V *is the constant vector field on* $\mathbb{C}^g$ *corresponding to the vector* V.

(*C*) *The equation*

$$\partial_V\left[\theta(Z+U)\,\theta(Z-U)\right]\partial_V\theta(Z) = \left[\theta(Z+U)\,\theta(Z-U)\right]\partial^2_{VV}\theta(Z) \pmod{\theta} \tag{1.7}$$

is valid on the theta-divisor $\Theta = \{Z \in X \mid \theta(Z) = 0\}$.

Equation (1.3) is one of the two auxiliary linear problems for the $2D$ Toda lattice equation, which can be regarded as a discretization of the KP equation. The idea to use it for the characterization of Jacobians was motivated by ([12]), and the author's earlier work with Zabrodin ([18]), where a connection of the theory of elliptic solutions of the $2D$ Toda lattice equations and the theory of the elliptic Ruijsenaars-Schneider system was established. In fact, Theorem 1.1 in a slightly different form was proved in ([18]) under the additional assumption that the vector U *spans an elliptic curve* in X.

The equivalence of (A) and (B) is a direct corollary of the addition formula for the theta-function. The statement (B) is the second particular case of the trisecant conjecture: the line in $\mathbb{CP}^{2^g-1}$ passing through the points $K((A-U)/2)$ and $K((A+U)/2)$ of the Kummer variety is tangent to $K(X)$ at the point $K((A-U)/2)$.

The "only if" part of (A) follows from the author's construction of solutions of the $2D$ Toda lattice equations [17]. The statement (C) is actually what we use for the proof of the theorem. It is stronger than (A). The implication $(A) \to (C)$ does not require the explicit theta-functional formula for ψ. It is enough to require only that equation (1.3) with u as in (1.4) has *local meromorphic* in x solutions which are holomorphic outside the divisor $\theta(Ux + Vt + Z) = 0$.

To put it more precisely, let $\tau(x,t)$ be a holomorphic function of x in some domain $\mathcal{D}$, where it has a simple root $\eta(t)$. If $\tau(\eta(t) \pm 1, t) \neq 0$, then the condition that equation (1.3)

with $u = (T-1)v$, where $v(x,t)) = -\partial_t \ln \tau(x,t)$, has a meromorphic solution with the only pole in $\mathcal{D}$ at η implies

$$\ddot{\eta} = \dot{\eta}\left[2v_0(t) - v(\eta+1,t) - v(\eta-1,t)\right] , \qquad (1.8)$$

where "dots" stands for the t-derivatives and v_0 is the coefficient of the Laurent expansion of $v(x,t)$ at η, i.e.

$$v(x,t) = \frac{\dot{\eta}}{x-\eta} + v_0(t) + O((x-\eta)). \qquad (1.9)$$

Formally, if we represent τ as an infinite product,

$$\tau(x,t) = c(t)\prod_i (x - x_i(t)), \qquad (1.10)$$

then equation (1.8) can be written as the infinite system of equations

$$\ddot{x}_i = \sum_{j\neq i} \dot{x}_i \dot{x}_j \left[\frac{2}{(x_i - x_j)} - \frac{1}{(x_i - x_j + 1)} - \frac{1}{(x_i - x_j - 1)}\right] . \qquad (1.11)$$

If τ is a rational, trigonometric or elliptic polynomial, then the system (1.11) coincides with the equations of motion for the rational, trigonometrical or elliptic Ruijsenaars-Schneider systems, respectively. Equations (1.11) are analogues of the equations derived in [15] and called in [12] the formal Calogero-Moser system.

Simple expansion of θ at the points of its divisor $Z \in \Theta : \theta(Z) = 0$ shows that for $\tau = \theta(Ux + Vt + Z)$ equation (1.8) is equivalent to (1.7).

The proof of the theorem goes along the same lines as the proof of Theorem 1.1 in [12]. In order to stress the similarity we almost literally copy some parts of [12]. At the beginning of the next section we derive equations (1.8) and show that they are sufficient conditions for the *local* existence of *formal wave solutions*. The formal wave solution of equation (1.3) is a solution of the form

$$\psi(x,t,k) = k^x e^{kt}\left(1 + \sum_{s=1}^{\infty} \xi_s(x,t)\, k^{-s}\right) . \qquad (1.12)$$

The ultimate goal is to show the existence of the wave solutions such that coefficients of the series (1.12) have the form $\xi_s = \xi_s(Ux + Vt + Z)$, where

$$\xi_s(Z) = \frac{\tau_s(Z)}{\theta(Z)} , \qquad (1.13)$$

and $\tau_s(Z)$ is a holomorphic function. The functions ξ_s are defined recursively by differential-difference equations $(T_U - 1)\,\xi_{s+1} = \partial_V \xi_s + u\xi_s$, where $T_U = e^{\partial_U}$ and ∂_U is a constant vector-field defined by the vector U.

In the case of differential equations the cohomological arguments, that are due to Lee-Oda-Yukie, can be applied for an attempt to glue local solutions into the global ones (see details in [3, 11]). These arguments were used in [3] and revealed that the core of the problem in the proof of Novikov's conjecture is a priori nontrivial cohomological obstruction for the

global solvability of the corresponding equations. The hardest part of the Shiota's work was the proof that the certain bad locus $\Sigma \subset \Theta$, which controls the obstruction, is empty [1].

In the difference case there is no analog of the cohomological arguments and we use a different approach. Instead of *proving* the global existence of solutions we, to some extend, *construct* them by defining first their residue on the theta-divisor. It turns out that the residue is regular on Θ outside the *singular locus* Σ which is the maximal T_U-invariant subset of Θ, i.e. $\Sigma = \bigcap_{k\in\mathbb{Z}} T_U^k\Theta$.

As in [12], we don't prove directly that the bad locus is empty. Our first step is to construct certain wave solutions outside the bad locus. We call them λ-periodic wave solutions. They are defined uniquely up to t-independent T_U-invariant factor. Then we show that for each $Z \notin \Sigma$ the λ-periodic wave solution is a common eigenfunction of a commutative ring $\mathcal{A}^Z$ of ordinary difference operators. The coefficients of these operators are independent of ambiguities in the construction of ψ. For the generic Z the ring $\mathcal{A}^Z$ is maximal and the corresponding spectral curve Γ is Z-independent. The correspondence $j : Z \longmapsto \mathcal{A}^Z$ and the results of the works [19, 20], where a theory of rank 1 commutative rings of difference operators was developed, allows us to make the next crucial step and prove the global existence of the wave function. Namely, on $(X \setminus \Sigma)$ the wave function can be globally defined as the preimage $j^*\psi_{BA}$ under j of the Baker-Akhiezer function on Γ and then can be extended on X by usual Hartogs' arguments. The global existence of the wave function implies that X contains an orbit of the KP hierarchy, as an abelian subvariety. The orbit is isomorphic to the generalized Jacobian $J(\Gamma) = \mathrm{Pic}^0(\Gamma)$ of the spectral curve ([3]). Therefore, the generalized Jacobian is compact. The compactness of $J(\Gamma)$ implies that the spectral curve is smooth and the correspondence j extends by linearity and defines the isomorphism $j : X \to J(\Gamma)$.

In the last section we present the proof of the last "fully discrete" case of the trisecant conjecture.

Theorem 1.2 *An indecomposable, principally polarized abelian variety (X,θ) is the Jacobian of a smooth curve of genus g if and only if there exist non-zero g-dimensional vectors $U \neq V \neq A \neq U \,(\mathrm{mod}\Lambda)$ such that one of the following equivalent conditions holds:*

(A) The difference equation

$$\psi(m, n+1) = \psi(m+1, n) + u(m, n)\psi(m, n) \tag{1.14}$$

is satisfied for

$$u(m, n) = \frac{\theta((m+1)U + (n+1)V + Z)\,\theta(mU + nV + Z)}{\theta(mU + (n+1)V + Z)\,\theta((m+1)U + nV + Z)} \tag{1.15}$$

and

$$\psi(m, n) = \frac{\theta(A + mU + nV + Z)}{\theta(mU + nV + Z)}\, e^{mp+nE}, \tag{1.16}$$

where p, E are constants and Z is arbitrary.

[1] The author is grateful to Enrico Arbarello for an explanation of these deep ideas and a crucial role of the singular locus Σ, which helped him to focus on the heart of the problem.

(*B*) *The equations*

$$\Theta[\varepsilon,0]\left(\frac{A-U-V}{2}\right)+e^{p}\Theta[\varepsilon,0]\left(\frac{A+U-V}{2}\right)=e^{E}\Theta[\varepsilon,0]\left(\frac{A+V-U}{2}\right), \tag{1.17}$$

are satisfied for all $\varepsilon\in\frac{1}{2}Z_2^g$.

(*C*) *The equation*

$$\theta(Z+U)\,\theta(Z-V)\;\theta(Z-U+V)+\theta(Z-U)\,\theta(Z+V)\;\theta(Z+U-V)=0\ (\mathrm{mod}\,\theta) \tag{1.18}$$

is valid on the theta-divisor $\Theta=\{Z\in X\mid\theta(Z)=0\}$.

Equation (1.14) is one of the two auxiliary linear problems for the so-called bilinear discrete Hirota equation (BDHE). The "only if" part of (*A*) follows from the author's work [21]. Under the assumption that the vector U *spans an elliptic curve* in X, theorem 1.2 was proved in [22], where the connection of the elliptic solutions of BDHE and the so-called elliptic nested Bete ansatz equations was established.

2 λ-periodic wave solutions

To begin with, let us show that equations (1.8) are necessary for the existence of a meromorphic solution of equation (1.3), which is holomorphic outside of the theta-divisor.

Let $\tau(x,t)$ be a holomorphic function of the variable x in some *translational invariant* domain $\mathcal{D}=T\mathcal{D}\subset\mathbb{C}$, where $T:x\to x+1$. We assume that τ is a smooth function of the parameter t. Suppose that τ in $\mathcal{D}$ has a simple root $\eta(t)$ such that

$$\tau(\eta(t)+1,t)\,\tau(\eta(t)-1,t)\neq 0. \tag{2.1}$$

Lemma 2.1 *If equation (1.3) with the potential* $u=(T-1)v$, *where* $v=-\partial_t\ln\tau(x,t)$ *has a meromorphic in* $\mathcal{D}$ *solution* $\psi(x,t)$, *with the simple pole at* $x=\eta$, *and regular at* $\eta-1$, *then equation (1.8) holds.*

Proof. Consider the Laurent expansions of ψ and v in the neighborhood of one of η:

$$v=\frac{\dot\eta}{x-\eta}+v_0+\dots\,, \tag{2.2}$$

$$\psi=\frac{\alpha}{x-\eta}+\beta+\dots\,. \tag{2.3}$$

(All coefficients in these expansions are smooth functions of the variable t). Substitution of (2.2,2.3) in (1.3) gives an infinite system of equations. We use only the following three of them.

The vanishing of the residue at η of the left hand side of (1.3) implies

$$\dot\alpha=\dot\eta\beta+\alpha(v_0-v(\eta+1,t)). \tag{2.4}$$

The vanishing of the residue and the constant terms of the Laurent expansion of (1.3) at $\eta - 1$ are equivalent to the equations

$$\alpha = \dot{\eta}\psi(\eta - 1, t)\,, \tag{2.5}$$

$$\partial_t \psi(\eta - 1, t) = \beta + [v(\eta - 1, t) - v_0]\psi(\eta - 1, t)\,. \tag{2.6}$$

Taking the t-derivative of (2.5) and using equations (2.4, 2.6) we get (1.8).

Let us show that equations (1.8) are sufficient for the existence of *local* meromorphic wave solutions which are holomorphic outside of the zeros of τ. In the difference case a notion of local solutions needs some clarification.

In the lemma below we assume that a translational invariant domain $\mathcal{D}$ is a disconnected union of small discs, i.e.

$$\mathcal{D} = \cup_{i\in\mathbb{Z}}\, T^i D_0, \quad D_0 = \{x \in \mathbb{C} \,|\, x - x_0| < 1/2\}. \tag{2.7}$$

Lemma 2.2 *Suppose that $\tau(x,t)$ is holomorphic in a domain $\mathcal{D}$ of the form (2.7) where it has simple zeros, for which condition (2.1) and equation (1.8) hold. Then there exist meromorphic wave solutions of equation (1.3) that have simple poles at zeros of τ and are holomorphic everywhere else.*

Proof. Substitution of (1.12) into (1.3) gives a recurrent system of equations

$$(T - 1)\xi_{s+1} = \dot{\xi}_s + u\xi_s. \tag{2.8}$$

Under the assumption that $\mathcal{D}$ is a disconnected union of small disks, ξ_{s+1} can be defined as an arbitrary meromorphic function in D_0 and then extended on $\mathcal{D}$ with the help of (2.8). If η is a zero of τ, then in this way we get a meromorphic function ξ_{s+1}, which a priori has poles at the points $\eta_k = \eta - k$ for all non-negative k. Our goal is to prove by induction that in fact (2.8) has meromorphic solutions with simple poles only at the zeros of τ.

Suppose that ξ_s has a simple pole at $x = \eta$

$$\xi_s = \frac{r_s}{x - \eta} + r_{s0} + r_{s1}(x - \eta) + \cdots\,. \tag{2.9}$$

The condition that ξ_{s+1} has no pole at $\eta - 1$ is equivalent to the equation

$$R_s = \dot{\eta}\xi_s(\eta - 1, t), \tag{2.10}$$

where by definition

$$R_s = r_s(v(\eta + 1, t) - v_0) + \dot{\eta} r_{s0} - \dot{r}_s\,. \tag{2.11}$$

Equation (2.10) with R_s given by (2.11) is our induction assumption. We need to show that the next equation holds also. From (2.8) it follows that

$$r_{s+1} = \dot{\eta}\xi_s(\eta - 1, t)\,, \tag{2.12}$$

$$\xi_{s+1}(\eta - 1, t) - r_{s+1,0} = -\dot{\xi}_s(\eta - 1, t)(v(\eta - 1, t) - v_0)\xi_s(\eta - 1, t)\,. \tag{2.13}$$

These equations imply

$$R_{s+1} = \dot{\eta}\xi_{s+1}(\eta-1,t) - (\ddot{\eta} + \dot{\eta}(v(\eta+1,t) + v(\eta-1,t) - 2v_0))\xi_s(\eta-1,t) \tag{2.14}$$

and the lemma is proved.

If ξ^0_{s+1} is a particular solution of (2.8), then the general solution is of the form $\xi_{s+1}(x,t) = c_{s+1}(x,t)+\xi^0_{s+1}(x,t)$ where $c_{s+1}(x,t)$ is T-invariant function of the variable x, and an *arbitrary* function of the variable t. Our next goal is to fix a *translation-invariant* normalization of ξ_s.

Let us show that in the periodic case $v(x+N,t) = v(x,t) = -\partial_t\tau(x,t)$, $N \in \mathbb{Z}$, the periodicity condition for $\xi_{s+1}(x+N,t) = \xi_{s+1}(x,t)$ uniquely defines t-dependence of the functions $c_s(x,t)$ (compare with the normalization of the Bloch solutions of differential equations used in [23]). Assume that ξ_{s-1} is known and satisfies the condition that there exists a periodic solution ξ^0_s of the corresponding equation. Let ξ^*_{s+1} be a solution of (2.8) for ξ^0_s. Then the function $\xi^0_{s+1} = \xi^*_{s+1} + x\partial_t c_s + c_s v$ is a solution of (2.8) for $\xi_s = \xi^0_s + c_s$. A choice of T-invariant function $c_s(x,t)$ does not affect the periodicity property of ξ_s, but it does affect the periodicity in x of the function $\xi^0_{s+1}(x,t)$. In order to make $\xi^0_{s+1}(x,t)$ periodic, the function $c_s(x,t)$ should satisfy the linear differential equation

$$N\partial_t c_s(x,t) + \xi^*_{s+1}(x+N,t) - \xi^*_{s+1}(x,t) = 0. \tag{2.15}$$

That defines $c_s(x,t)$ uniquely up to a t-independent T-invariant function of the variable x.

In the general case, when U is not a point of finite order in X, the solution of the normalization problem for the coefficients of the wave solutions requires the global existence of these coefficients along certain affine subspaces.

Let Y_U be the Zariski closure of the group $\{Uk \mid k \in \mathbb{Z}\}$ in X. As an abelian subvariety, it is generated by its irreducible component Y^0_U, containing 0, and by the point U_0 of finite order in X, such that $U - U_0 \in Y^0_U$, $NU_0 = \lambda_0 \in \Lambda$. Shifting Z if needed, we may assume, without loss of generality, that 0 is not in the singular locus $T_U\Sigma = \Sigma \subset \Theta$. Then $Y_U \cap \Sigma = \emptyset$, because any T_U-invariant set is Zariski dense in Y_U. Note, that for *sufficiently small* t *the affine subvariety* $Y_U + Vt$ *does not intersect* Σ*, as well.*

Consider the restriction of the theta-function onto the subspace $\mathcal{C} + Vt \subset \mathbb{C}^g$:

$$\tau(z,t) = \theta(z + Vt), \quad z \in \mathcal{C}. \tag{2.16}$$

Here and below $\mathcal{C}$ is a union of affine subspaces, $\mathcal{C} = \cup_{r\in\mathbb{Z}}(\mathbb{C}^d + rU_0)$, where $\mathbb{C}^d$ is a linear subspace that is the irreducible component of $\pi^{-1}(Y^0_U)$, and $\pi : \mathbb{C}^g \to X = \mathbb{C}^g/\Lambda$ is the universal cover of X.

The restriction of equation (1.7) onto Y_U gives the equation

$$\partial_t\left(\tau(z+U,t)\,\tau(z-U,t)\right)\,\partial_t\tau(z,t) = \tau(z+U,t)\,\tau(z-U,t)\,\partial^2_{tt}\tau(z,t) \pmod{\tau}, \tag{2.17}$$

which is valid on the divisor $\mathcal{T}^t = \{z \in \mathcal{C} \mid \tau(z,t) = 0\}$. For fixed t, the function $u(z,t)$ has simple poles on the divisors $\mathcal{T}^t$ and $\mathcal{T}^t_U = \mathcal{T}^t - U$.

Lemma 2.3 *Let equation (2.17) for $\tau(z,t)$ holds and let $\lambda_1,\ldots,\lambda_d$ be a set of linear independent vectors of the sublattice $\Lambda_U^0=\Lambda\cap\mathbb{C}^d\subset\mathbb{C}^g$. Then for sufficiently small t equation (1.3) with the potential $u(Ux+z,t)$, restricted to $x=n\in\mathbb{Z}$, has a unique, up to a z-independent factor, wave solution of the form $\psi=k^x e^{kt}\phi(Ux+z,t,k)$ such that:*

(i) the coefficients $\xi_s(z,t)$ of the formal series

$$\phi(z,t,k)=e^{bt}\left(1+\sum_{s=1}^{\infty}\xi_s(z,t)\,k^{-s}\right) \tag{2.18}$$

are meromorphic functions of the variable $z\in\mathbb{C}^d$ with a simple pole at the divisor $\mathcal{T}^t$,

$$\xi_s(z,t)=\frac{\tau_s(z,t)}{\tau(z,t)}\,; \tag{2.19}$$

(ii) $\phi(z,t,k)$ is quasi-periodic with respect to the lattice Λ_U

$$\phi(z+\lambda,t,k)=\phi(z,t,k)\,B^{\lambda}(k),\quad \lambda\in\Lambda_U; \tag{2.20}$$

and is periodic with respect to the vectors $\lambda_0,\lambda_1,\ldots,\lambda_d$, i.e.,

$$B^{\lambda_i}(k)=1,\quad i=0,\ldots,d. \tag{2.21}$$

Proof. The functions $\xi_s(z)$ are defined recursively by the equations

$$\Delta_U\,\xi_{s+1}=\dot{\xi}_s+(u+b)\,\xi_s. \tag{2.22}$$

Here and below Δ_U stands for the difference derivative $e^{\partial_U}-1$. The quasi-periodicity conditions (2.20) for ϕ are equivalent to the equations

$$\xi_s(z+\lambda,t)-\xi_s(z,t)=\sum_{i=1}^{s}B_i^{\lambda}\xi_{s-i}(z,t)\,,\quad \xi_0=1. \tag{2.23}$$

A particular solution of the first equation $\Delta_U\,\xi_1=u+b$ is given by the formula

$$\xi_1^0=-\partial_V\ln\theta+l_1(z)\,b, \tag{2.24}$$

where $l_1(z)$ is a linear form on $\mathcal{C}$ such that $l_1(U)=1$. It satisfies the monodromy relations (2.23) with

$$B_1^{\lambda}=l_1(\lambda)\,b-\partial_V\ln\theta(z+\lambda)+\partial_V\ln\theta(z)\,, \tag{2.25}$$

If $U_0\neq 0$, then the space of linear forms on $\mathcal{C}$ is $(d+1)$-dimensional. Therefore, the equation $l_1(U)=1$ and $(d+1)$ normalization conditions $B_1^{\lambda_i}=1$, $i=0,1,\ldots,d$, defines uniquely the constant b, the form l_1, and then the constants B_1^{λ} for all $\lambda\in\Lambda_U$. Note, that if $U_0=0$, then the space of linear forms on $\mathbb{C}^d$ is d-dimensional, but the normalization condition $B_1^{\lambda_0}$ becomes trivial.

Let us assume that the coefficient ξ_{s-1} of the series (2.18) is known, and that there exists a solution ξ_s^0 of the next equation, which is holomorphic outside of the divisor $\mathcal{T}^t$, and which

satisfies the quasi-periodicity conditions (2.23). We assume also that ξ_s^0 is unique up to the transformation $\xi_s = \xi_s^0 + c_s(t)$, where $c_s(t)$ is a time-dependent constant. As it was shown above, the induction assumption holds for $s = 1$.

Let us define a function $\tau_{s+1}^0(z)$ on $\mathcal{T}^t$ with the help of the formula

$$\tau_{s+1}^0 = -\partial_t \tau_s(z,t) - b\tau_s(z,t) + \frac{\partial_t \tau(z+U,t)}{\tau(z+U,t)} \tau_s(z,t), \quad z \in \mathcal{T}^t. \tag{2.26}$$

where $\tau_s = \theta(Ux+Vt+z)\xi_s$. Note, that the restriction of τ_s on $\mathcal{T}^t$ does not depend on $c_s(t)$. Simple expansion of τ and τ_s at the generic point of $\mathcal{T}^t$ shows that the residue in U-line of the right hand side of (2.26) is equal to R_{s+1}, where R_s is defined by (2.11). The induction assumption (2.10) of Lemma 2.2 is equivalent to the statement: *if ξ_s is a solution of equation (2.22) for $s-1$, then the function τ_{s+1}^0, given by (2.26) is equal to*

$$\tau_{s+1}^0 = -\partial_t \tau(z,t) \frac{\tau_s(z-U,t)}{\tau(z-U,t)}, \quad z \in \mathcal{T}^t. \tag{2.27}$$

Let us show that τ_{s+1}^0 is holomorphic on $\mathcal{T}^t$. Equations (2.26) and (2.27) imply that τ_{s+1}^0 is regular on $\mathcal{T}^t \setminus \Sigma_*$, where $\Sigma_* = \mathcal{T}^t \cap \mathcal{T}_U^t \cap \mathcal{T}_{-U}^t$. Let z_0 be a point of Σ_*. By the assumption, Y_U does not intersect Σ. Hence, $\mathcal{T}^t$, for sufficiently small t, does not intersect Σ, as well. Therefore, there exists an integer $k > 0$ such that $z_k = z_0 - kU$ is in $\mathcal{T}^t$, and $\tau(z_{k+1}, t) \neq 0$. Then, from equation (2.27) it follows that τ_{s+1}^0 is regular at the point $z = z_k$. Using equation (2.26) for $z = z_k$, we get that $\partial_t \tau(z_{k-1}, t)\tau_s(z_k, t) = 0$. The last equality and the equation (2.27) for $z = z_{k-1}$ imply that τ_{s+1}^0 is regular at the point z_{k-1}. Regularity of τ_{s+1}^0 at z_{k-1} and equation (2.26) for $z = z_{k-1}$ imply $\partial_t \tau(z_{k-2}, t)\tau_s(z_{k-1}, t) = 0$. Then equation (2.27) for $z = z_{k-2}$ implies that τ_{s+1}^0 is regular at the point z_{k-2}. By continuing these steps we get finally that τ_{s+1}^0 is regular at $z = z_0$. Therefore, τ_{s+1}^0 is regular on $\mathcal{T}^t$.

Recall, that an analytic function on an analytic divisor in $\mathbb{C}^d$ has a holomorphic extension onto $\mathbb{C}^d$ ([24]). The space $\mathcal{C}$ is a union of affine subspaces. Therefore, there exists a holomorphic function $\tau^*(z,t), z \in \mathcal{C}$, such that $\tau_{s+1}^*|_{\mathcal{T}^t} = \tau_{s+1}^0$. Consider the function $\chi_{s+1}^* = \tau_{s+1}^*/\tau$. It is holomorphic outside of the divisor $\mathcal{T}^t$. From (2.23) it follows that it satisfies the relations

$$\chi_{s+1}^*(z+\lambda) - \chi_{s+1}^*(z) = f_{s+1}^\lambda(z) + \sum_{i=1}^{s} B_i^\lambda \xi_{s+1-i}(z,t)\,, \tag{2.28}$$

where $f_{s+1}^\lambda(z)$ is a holomorphic function of $z \in \mathcal{C}$. It satisfies the twisted homomorphism relations

$$f_{s+1}^{\lambda+\mu}(z) = f_{s+1}^\lambda(z+\mu) + f_{s+1}^\mu(z), \tag{2.29}$$

i.e., it defines an element of the first cohomology group of Λ_U with coefficients in the sheaf of holomorphic functions, $f \in H^1_{gr}(\Lambda_U, H^0(\mathcal{C}, \mathcal{O}))$. The same arguments, as that used in the proof of the part (b) of the Lemma 12 in [3], show that there exists a holomorphic function $h_{s+1}(z)$ such that

$$f_{s+1}^\lambda(z) = h_{s+1}(z+\lambda) - h_{s+1}(z) + \tilde{B}_{s+1}^\lambda, \tag{2.30}$$

where $\tilde{B}^{\lambda}_{s+1}$ is a constant. Hence, the function $\chi_{s+1} = \chi^*_{s+1} + h$ has the following monodromy properties

$$\chi_{s+1}(z+\lambda) - \chi_{s+1}(z) = \tilde{B}^{\lambda}_{s+1} + \sum_{i=1}^{s} B_i^{\lambda} \xi_{s+1-i}(z,t), \tag{2.31}$$

Let us try to find a solution of (2.22) in the form $\xi^0_{s+1} = \chi_{s+1} + \zeta_{s+1}$. That gives us the equation

$$\Delta_U \zeta_{s+1} = g_s, \tag{2.32}$$

where

$$g_s = -\Delta_U\, \chi_{s+1} + \dot{\xi}^0_s + (u+b)\xi^0_s + \partial_t c_s + (u+b)c_s. \tag{2.33}$$

From (2.23,2.31) it follows that g_s is periodic with respect to the lattice Λ_U, i.e., it is a function on Y_U. Equation (2.28) and the statement of Lemma 2.2 implies that it is a holomorphic function. Therefore, g_s is constant on each of the irreducible components of $\mathcal{C}$:

$$g_s(z_0 + rU_0) = g_s^{(r)}, \quad z_0 \in \mathbb{C}^d \subset \mathcal{C}. \tag{2.34}$$

Hence, the general solution of equation (2.32), such that the corresponding solution ξ_{s+1} of (2.22) satisfies the quasi-periodicity conditions, is given by the formula

$$\zeta_{s+1}(z_0 + rU_0) = l_{s+1}(z_0) + \sum_{i=0}^{r-1} (g_s^{(i)} - a_s) + c_{s+1}, \tag{2.35}$$

where $l_{s+1}(z_0)$ is a linear form on $\mathbb{C}^d$; the constant a_s equals $a_s = l_{s+1}(U - U_0)$. The normalization condition (2.21) for $B^{\lambda_i}_{s+1} = 1, i = 0, \ldots, d$ defines uniquely l_{s+1} and $\partial_t c_s$, i.e. the time-dependence of $c_s(t)$. The induction step is completed and thus the lemma is proven.

Note, that a simple shift $z \to z + Z$, where $Z \notin \Sigma$, gives λ-periodic wave solutions with meromorphic coefficients along the affine subspaces $Z + \mathcal{C}$. These wave solutions are related to each other by constant factors. Therefore choosing, in the neighborhood of any $Z \notin \Sigma$, a hyperplane orthogonal to the vector U and fixing initial data on this hyperplane at $t = 0$, we define the corresponding series $\phi(z + Z, t, k)$ as a *local* meromorphic function of Z and the *global* meromorphic function of $z \in \mathcal{C}$.

3 Commuting difference operators

In this section we show that λ-periodic wave solutions of equation (1.3), constructed in the previous section, are common eigenfunctions of rings of commuting difference operators.

Lemma 3.1 *Let the assumptions of Theorem 1.1 hold. Then, there is a unique pseudo-difference operator*

$$\mathcal{L}(Z) = T + \sum_{s=0}^{\infty} w_s(Z) T^{-s} \tag{3.1}$$

such that

$$\mathcal{L}(Ux + Vt + Z)\, \psi = k\, \psi\,, \tag{3.2}$$

where $\psi = k^x e^{kt}\phi(Ux + Z, t, k)$ is a λ-periodic wave solution of (1.3). The coefficients $w_s(Z)$ of $\mathcal{L}$ are meromorphic functions on the abelian variety X with poles along the divisors $T_U^{-i}\Theta = \Theta - iU,\ i \le s$.

Proof. The construction of $\mathcal{L}$ is standard for the theory of $2D$ Toda lattice equations. First we define $\mathcal{L}$ as a pseudo-difference operator with coefficients $w_s(Z,t)$, which are functions of Z and t.

Let ψ be a λ-periodic wave solution. The substitution of (2.18) in (3.2) gives a system of equations that recursively define $w_s(Z,t)$, as difference polynomials in the coefficients of ψ. The coefficients of ψ are local meromorphic functions of Z, but the coefficients of $\mathcal{L}$ are well-defined *global meromorphic functions* of on $\mathbb{C}^g \setminus \Sigma$, because different λ-periodic wave solutions are related to each other by a factor, which does not affect $\mathcal{L}$. The singular locus is of codimension ≥ 2. Then Hartogs' holomorphic extension theorem implies that $w_s(Z,t)$ can be extended to a global meromorphic function on $\mathbb{C}^g$.

The translational invariance of u implies the translational invariance of $\mathcal{L}$. Indeed, for any constant s the series $\phi(Vs + Z, t-s, k)$ and $\phi(Z,t,k)$ correspond to λ-periodic solutions of the same equation. Therefore, they coincide up to a T_U-invariant factor. This factor does not affect $\mathcal{L}$. Hence, $w_s(Z,t) = w_s(Vt + Z)$.

For any $\lambda' \in \Lambda$, the λ-periodic wave functions corresponding to Z and $Z + \lambda'$ are also related to each other by a T_U-invariant factor. Hence, w_s are periodic with respect to Λ, and therefore, are meromorphic functions on the abelian variety X. The lemma is proved.

Consider now the strictly positive difference parts of the operators $\mathcal{L}^m$. Let $\mathcal{L}^m_+$ be the difference operator such that $\mathcal{L}^m_- = \mathcal{L}^m - \mathcal{L}^m_+ = F_m + F^1_m T^{-1} + O(T^{-2})$. By definition the leading coefficient F_m of $\mathcal{L}^m_-$ is the residue of $\mathcal{L}^m$:

$$F_m = \operatorname{res}_T \mathcal{L}^m,\ F^1_m = \operatorname{res}_T \mathcal{L}^m\, T. \tag{3.3}$$

From the construction of $\mathcal{L}$ it follows that $[\partial_t - T + u, \mathcal{L}^n] = 0$. Hence,

$$[\partial_t - T + u, \mathcal{L}^m_+] = -[\partial_t - T + u, \mathcal{L}^m_-] = (\Delta_U F_m)\, T. \tag{3.4}$$

Indeed, the left hand side of (3.4) shows that the right hand side is a difference operator with non-vanishing coefficients only at the positive powers of T. The intermediate equality shows that this operator is at most of order 1. Therefore, it has the form $f_m T$. The coefficient f_m is easy expressed in terms of the leading coefficient $\mathcal{L}^m_-$. Note, that the vanishing of the coefficient at T^0 implies the equation

$$\partial_V F_m = \Delta_U\, F^1_m, \tag{3.5}$$

which we will use later.

The functions $F_m(Z)$ are difference polynomials in the coefficients w_s of $\mathcal{L}$. Hence, $F_m(Z)$ are meromorphic functions on X. Next statement is crucial for the proof of the existence of commuting differential operators associated with u.

Lemma 3.2 *The abelian functions F_m have at most simple poles on the divisors Θ and Θ_U.*

Proof. We need a few more standard constructions from $2D$ Toda theory. If ψ is as in Lemma 3.1, then there exists a unique pseudo-difference operator Φ such that

$$\psi = \Phi k^x e^{kt}, \quad \Phi = 1 + \sum_{s=1}^{\infty} \varphi_s(Ux + Z, t) T^{-s}. \tag{3.6}$$

The coefficients of Φ are universal difference polynomials in ξ_s. Therefore, $\varphi_s(z + Z, t)$ is a global meromorphic function of $z \in \mathcal{C}$ and a local meromorphic function of $Z \notin \Sigma$. Note, that $\mathcal{L} = \Phi T \Phi^{-1}$.

Consider the dual wave function defined by the left action of the operator Φ^{-1}: $\psi^+ = \left(k^{-x} e^{-kt}\right) \Phi^{-1}$. Recall that the left action of a pseudo-difference operator is the formal adjoint action under which the left action of T on a function f is $(fT) = T^{-1} f$. If ψ is a formal wave solution of (1.3), then ψ^+ is a solution of the adjoint equation

$$\left(-\partial_t - T^{-1} + u\right) \psi^+ = 0. \tag{3.7}$$

The same arguments, as before, prove that if equations (1.8) for poles of v hold then ξ_s^+ have simple poles at the poles of Tv. Therefore, if ψ as in Lemma 2.3, then the dual wave solution is of the form $\psi^+ = k^{-x} e^{kt} \phi^+(Ux + Z, t, k)$, where the coefficients $\xi_s^+(z + Z, t)$ of the formal series

$$\phi^+(z + Z, t, k) = e^{-bt} \left(1 + \sum_{s=1}^{\infty} \xi_s^+(z + Z, t)\, k^{-s}\right) \tag{3.8}$$

are λ-periodic meromorphic functions of the variable $z \in \mathcal{C}$ with the simple pole at the divisor $T_U^{-1} \mathcal{T}^t$.

The ambiguity in the definition of ψ does not affect the product

$$\psi^+ \psi = \left(k^{-x} e^{-kt} \Phi^{-1}\right) \left(\Phi k^x e^{kt}\right). \tag{3.9}$$

Therefore, although each factor is only a local meromorphic function on $\mathbb{C}^g \backslash \Sigma$, the coefficients J_s of the product

$$\psi^+ \psi = \phi^+(Z, t, k)\, \phi(Z, t, k) = 1 + \sum_{s=1}^{\infty} J_s(Z, t)\, k^{-s} \tag{3.10}$$

are *global meromorphic functions* of Z. Moreover, the translational invariance of u implies that they have the form $J_s(Z, t) = J_s(Z + Vt)$. The factors in the left hand side of (3.10) have the simple poles on $\Theta - Vt$ and $\Theta - U - Vt$. Hence, $J_s(Z)$ is a meromorphic function on X with the simple poles at Θ and $T_U^{-1} \Theta = \Theta_U$.

From the definition of $\mathcal{L}$ it follows that

$$\operatorname{res}_k \left(\psi^+ (\mathcal{L}^n \psi)\right) k^{-1} dk = \operatorname{res}_k \left(\psi^+ k^n \psi\right) k^{-1} dk = J_n. \tag{3.11}$$

On the other hand, using the identity

$$\operatorname{res}_k \left(k^{-x} \mathcal{D}_1\right) \left(\mathcal{D}_2 k^x\right) k^{-1} dk = \operatorname{res}_T \left(\mathcal{D}_2 \mathcal{D}_1\right), \tag{3.12}$$

we get

$$\operatorname{res}_k(\psi^+\mathcal{L}^n\psi)k^{-1}dk = \operatorname{res}_k\left(k^{-x}\Phi^{-1}\right)\left(\mathcal{L}^n\Phi k^x\right)k^{-1}dk = \operatorname{res}_T\mathcal{L}^n = F_n. \tag{3.13}$$

Therefore, $F_n = J_n$ and the lemma is proved.

Let $\hat{\mathbf{F}}$ be a linear space generated by $\{F_m,\ m = 1,\ldots\}$. It is a subspace of the 2^g-dimensional space of the abelian functions that have at most simple poles at Θ and Θ_U. Therefore, for all but $\hat{g} = \dim \hat{\mathbf{F}}$ positive integers n, there exist constants $c_{i,n}$ such that

$$F_n(Z) + \sum_{i=1}^{n-1} c_{i,n}F_i(Z) = 0. \tag{3.14}$$

Let I denote the subset of integers n for which there are no such constants. We call this subset the gap sequence.

Lemma 3.3 *Let $\mathcal{L}$ be the pseudo-difference operator corresponding to a λ-periodic wave function ψ constructed above. Then, for the difference operators*

$$L_n = \mathcal{L}_+^n + \sum_{i=1}^{n-1} c_{i,n}\mathcal{L}_+^{n-i} = 0,\ n \notin I, \tag{3.15}$$

the equations

$$L_n\,\psi = a_n(k)\,\psi, \quad a_n(k) = k^n + \sum_{s=1}^{\infty} a_{s,n}k^{n-s} \tag{3.16}$$

where $a_{s,n}$ are constants, hold.

Proof. First note that from (3.4) it follows that

$$[\partial_t - T + u, L_n] = 0. \tag{3.17}$$

Hence, if ψ is a λ-periodic wave solution of (1.3) corresponding to $Z \notin \Sigma$, then $L_n\psi$ is also a λ-periodic solution of the same equation. That implies the equation $L_n\psi = a_n(Z,k)\psi$, where a is T_U-invariant. The ambiguity in the definition of ψ does not affect a_n. Therefore, the coefficients of a_n are well-defined *global* meromorphic functions on $\mathbb{C}^g\setminus\Sigma$. The ∂_U- invariance of a_n implies that a_n, as a function of Z, is holomorphic outside of the locus. Hence it has an extension to a holomorphic function on $\mathbb{C}^g$. It is periodic with respect to the lattice Λ. Hence a_n is Z-independent. Note that $a_{s,n} = c_{s,n},\ s \le n$. The lemma is proved.

The operator L_m restricted to the points $x = n$ can be regarded as a Z-parametric family of ordinary difference operators L_m^Z, whose coefficients have the form

$$L_m^Z = T^m + \sum_{i=1}^{m-1} u_{i,m}(Un+Z)\,T^{m-i}, \quad m \notin I. \tag{3.18}$$

Corollary 3.1 *The operators L_m^Z commute with each other,*

$$[L_n^Z, L_m^Z] = 0, \quad Z \notin \Sigma. \tag{3.19}$$

From (3.16) it follows that $[L_n^Z, L_m^Z]\psi = 0$. The commutator is an ordinary difference operator. Hence, the last equation implies (3.19).

4 The spectral curve.

A theory of commuting difference operators containing a pair of operators of co-prime orders was developed in ([19, 20]). It is analogous to the theory of rank 1 commuting differential operators ([7, 8, 9, 10, 19]). (Relatively recently this theory was generalized to the case of commuting difference operators of arbitrary rank in [25].)

Lemma 4.1 *Let $\mathcal{A}^Z$, $Z \notin \Sigma$, be a commutative ring of ordinary difference operators spanned by the operators L_n^Z. Then there is an irreducible algebraic curve Γ of arithmetic genus $\hat{g} = \dim \hat{\mathbf{F}}$, such that for a generic Z the ring $\mathcal{A}^Z$ is isomorphic to the ring $A(\Gamma, P_+, P_-)$ of the meromorphic functions on Γ with the only pole at a smooth point P_+, vanishing at another smooth point P_-. The correspondence $Z \to \mathcal{A}^Z$ defines a holomorphic map of $X \setminus \Sigma$ to the space of torsion-free rank 1 sheaves $\mathcal{F}$ on Γ*

$$j : X \backslash \Sigma \longmapsto \overline{\mathrm{Pic}}(\Gamma). \tag{4.1}$$

Proof. As shown in ([19, 20]) there is a natural correspondence

$$\mathcal{A} \longleftrightarrow \{\Gamma, P_\pm, \mathcal{F}\} \tag{4.2}$$

between commutative rings $\mathcal{A}$ of ordinary linear difference operators containing a pair of monic operators of co-prime orders, and sets of algebraic-geometrical data $\{\Gamma, P_\pm, [k^{-1}]_1, \mathcal{F}\}$, where Γ is an algebraic curve with a fixed first jet $[k^{-1}]_1$ of a local coordinate k^{-1} in the neighborhood of a smooth point $P_+ \in \Gamma$ and $\mathcal{F}$ is a torsion-free rank 1 sheaf on Γ such that

$$h^0(\Gamma, \mathcal{F}(nP_+ - nP_-)) = h^1(\Gamma, \mathcal{F}(nP_+ - nP_-) = 0. \tag{4.3}$$

The correspondence becomes one-to-one if the rings $\mathcal{A}$ are considered modulo conjugation $\mathcal{A}' = g(x)\mathcal{A}g^{-1}(x)$.

The construction of the correspondence (4.2) depends on a choice of initial point $x_0 = 0$. The spectral curve and the sheaf $\mathcal{F}$ are defined by the evaluations of the coefficients of generators of $\mathcal{A}$ at a finite number of points of the form $x_0 + n$. In fact, the spectral curve is independent on the choice of x_0, but the sheaf does depend on it, i.e. $\mathcal{F} = \mathcal{F}_{x_0}$.

Using the shift of the initial point it is easy to show that the correspondence (4.2) extends to the commutative rings of operators whose coefficients are *meromorphic* functions of x. The rings of operators having poles at $x = 0$ correspond to sheaves for which the condition (4.3) for $n = 0$ is violated.

The algebraic curve Γ is called the spectral curve of $\mathcal{A}$. The ring $\mathcal{A}$ is isomorphic to the ring $A(\Gamma, P_+, P_-)$ of meromorphic functions on Γ with the only pole at the puncture P_+ and which vanish at P_-. The isomorphism is defined by the equation

$$L_a\psi_0 = a\psi_0, \quad L_a \in \mathcal{A}, \ a \in A(\Gamma, P_+, P_-). \tag{4.4}$$

Here ψ_0 is a common eigenfunction of the commuting operators. At $x = 0$ it is a section of the sheaf $\mathcal{F} \otimes \mathcal{O}(P_+)$.

Let Γ^Z be the spectral curve corresponding to $\mathcal{A}^Z$. It is well-defined for all $Z \notin \Sigma$. The eigenvalues $a_n(k)$ of the operators L_n^Z defined in (3.16) coincide with the Laurent expansions at P_+ of the meromorphic functions $a_n \in A(\Gamma^Z, P_\pm)$. They are Z-independent. Hence, the spectral curve is Z-independent, as well, $\Gamma = \Gamma^Z$. The first statement of the lemma is thus proven.

The construction of the correspondence (4.2) implies that if the coefficients of operators $\mathcal{A}$ holomorphically depend on parameters then the algebraic-geometrical spectral data are also holomorphic functions of the parameters. Hence j is holomorphic away of Θ. Then using the shift of the initial point and the fact, that $\mathcal{F}_{x_0}$ holomorphically depends on x_0, we get that j holomorphically extends on $\Theta \setminus \Sigma$, as well. The lemma is proved.

Remark. Recall, that a commutative ring $\mathcal{A}$ of linear ordinary difference operators is called maximal if it is not contained in any bigger commutative ring. As in the differential case (see details in [12]), it is easy to show that for the generic Z *the ring* $\mathcal{A}^Z$ *is maximal.*

Our next goal is to prove finally the global existence of the wave function.

Lemma 4.2 *Let the assumptions of the Theorem 1.1 hold. Then there exists a common eigenfunction of the operators* L_n^Z *of the form* $\psi = e^{kx}\phi(Ux + Z, k)$ *such that the coefficients of the formal series*

$$\phi(Z,k) = 1 + \sum_{s=1}^{\infty} \xi_s(Z)\, k^{-s} \tag{4.5}$$

are global meromorphic functions with a simple pole at Θ.

Proof. It is instructive to consider first the case when the spectral curve Γ of the rings $\mathcal{A}^Z$ is smooth. Then, as shown in ([20]), the corresponding common eigenfunction of the commuting differential operators (the Baker-Akhiezer function), normalized by the condition $\psi_0|_{x=0} = 1$, is of the form

$$\hat{\psi}_0 = \frac{\hat{\theta}(\hat{A}(P) + \hat{U}x + \hat{Z})\,\hat{\theta}(\hat{Z})}{\hat{\theta}(\hat{U}x + \hat{Z})\,\hat{\theta}(\hat{A}(P) + \hat{Z})}\, e^{x\,\Omega(P)}. \tag{4.6}$$

Here $\hat{\theta}(\hat{Z})$ is the Riemann theta-function constructed with the help of the matrix of b-periods of normalized holomorphic differentials on Γ; $\hat{A} : \Gamma \to J(\Gamma)$is the Abel map; Ω is the abelian integral corresponding to the third kind meromorphic differential $d\Omega$ with the residues ± 1 at the punctures $P_\pm$ and $2\pi i\hat{U}$ is the vector of its b-periods.

Remark. Let us emphasize, that the formula (4.6) is not the result of a solution of some difference equations. It is a direct corollary of analytic properties of the Baker-Akhiezer function $\hat{\psi}_0(x, P)$ on the spectral curve:

(i) $\hat{\psi}_0$ *is meromorphic function on the universal cover* $\tilde{\Gamma}$ *of* $\{\Gamma \setminus P_\pm\}$ *with the monodromy around* $P_\pm$ *equals* $e^{\pm 2\pi i x}$*; the pole divisor od* $\hat{\psi}_0$ *is of degree* $\tilde{g}$ *and is* x*-independent. It is non-special, if the operators are regular at the normalization point* $x = 0$;

(ii) in the neighborhood of P_0 *the function* $\hat{\psi}_0$ *has the form (1.12) (with* $t = 0$*).*

From the Riemann-Rokh theorem it follows that, if $\hat\psi_0$ exists, then it is unique. It is easy to check that the function $\hat\psi_0$ given by (4.6) has all the desired properties.

The last factors in the numerator and the denominator of (4.6) are x-independent. Therefore, the function

$$\hat\psi_{BA} = \frac{\hat\theta(\hat A(P) + \hat U x + \hat Z)}{\hat\theta(\hat U x + \hat Z)}\, e^{x\,\Omega(P)} \tag{4.7}$$

is also a common eigenfunction of the commuting operators.

In the neighborhood of P_+ the function $\hat\psi_{BA}$ has the form

$$\hat\psi_{BA} = k^x \left(1 + \sum_{s=1}^{\infty} \frac{\tau_s(\hat Z + \hat U x)}{\hat\theta(\hat U x + \hat Z)}\, k^{-s}\right), \quad k = e^{\Omega}, \tag{4.8}$$

where $\tau_s(\hat Z)$ are global holomorphic functions.

According to Lemma 4.1, we have a holomorphic map $\hat Z = j(Z)$ of $X \setminus \Sigma$ into $J(\Gamma)$. Consider the formal series $\psi = j^*\hat\psi_{BA}$. It is globally well-defined out of Σ. If $Z \notin \Theta$, then $j(Z) \notin \hat\Theta$ (which is the divisor on which the condition (4.3) is violated). Hence, the coefficients of ψ are regular out of Θ. The singular locus is at least of codimension 2. Hence, using once again Hartogs' arguments we can extend ψ on X.

If the spectral curve is singular, we can proceed along the same lines using a proper generalization of (4.7). Note, that in ([12]) we used the generalization of (4.7) given by the theory of Sato τ-function ([26]). In fact the general theory of the tau-function is not needed for our purposes. It is enough to consider only algebro-geometric points of the Sato Grassmanian.

Let $p : \hat\Gamma \longmapsto \Gamma$ be the normalization map, i.e. a regular map of a smooth genus $\tilde g$ algebraic curve $\hat\Gamma$ to the spectral curve Γ which is one-to-one outside the preimages $\hat q_k$ of singular points Γ. The normalized common eigenfunction $\hat\psi_0$ of commuting operators can be regarded as a multi-valued meromorphic function on $\hat\Gamma \setminus P_\pm$ with the monodromy $e^{\pm 2\pi i x}$ around the punctures $P_\pm$. The divisor $D = \sum_s \gamma_s$ of the poles of $\hat\psi_0$ is of degree $\tilde g + d \leq \hat g$, where $\hat g$ is the arithmetic genus of Γ. The expansions of $\hat\psi$ at the points $\hat q_k$ are in some linear subspace of co-dimension d in the space $\bigoplus_k \mathcal{O}_{\hat q_k}$. If we fix local coordinates z_k in the neighborhoods of $\hat q_k$, then the latter condition can be written as a system of n linear constraints

$$\sum_{k,j} c^i_{k,s} \partial^s_{z_k} \hat\psi_0|_{q_k} = 0, \quad i = 1, \ldots, d. \tag{4.9}$$

The constants $\hat c^i_{k,j}$ are defined up to the transformations $\hat c^i_{k,s} \to \sum_i g_i^{i'} \hat c^i_{k,s}$, where $g_i^{i'}$ is a non-degenerate matrix.

The analytical properties of the function $\hat\psi_0$ imply that it can be represented in the form

$$\hat\psi_0 = \sum_{i=0}^{d} r_i(x, D) \frac{\hat\theta(\hat A(P) + \hat U x + \hat Z_i)}{\hat\theta(\hat A(P) + \hat Z_i)}\, e^{x\,\Omega(P)}. \tag{4.10}$$

Here $Z_i = Z_* - \hat A(\gamma_{\tilde g + i})$, where $Z_* = R - \sum_{s=1}^{\tilde g - 1} \hat A(\gamma_s)$ and R is the vector of the Riemann constants.

The coefficients r_i in (4.10) are defined by the linear equations (4.9) and the normalization of the leading term in the expansion (1.12) of $\hat{\psi}_0$ at P_+. Keeping track of the x-dependent terms one can write the eigenfunction of the commuting operators in the form

$$\hat{\psi}_{BA} = \sum_{i=0}^{n} R_i \,\hat{\theta}(\hat{A}(P) + \hat{U}x + \hat{Z}_i), \tag{4.11}$$

where the coefficients R_i depend on x, but are P-independent. The equations (4.9) imply

$$\sum_{j=0}^{n} M_{i,j} R_j = 0, \tag{4.12}$$

where the entries of $(d \times (d+1))$-dimensional matrix M_{ij} are equal to

$$M_{ij} = \sum_{k,s} C^i_{k,s} \partial^s_{z_k} \hat{\theta}(\hat{A}(P(z_k)) + \hat{U}x + \hat{Z}_j)|_{z_k=0}. \tag{4.13}$$

The coefficients $C^j_{k,s}$ in (4.13) can be expressed in terms of D and the coefficients $c^i_{k,s}$ in (4.9). They and the divisor D can be regarded as parameters defining the sheaf $\mathcal{F}$ in (4.2).

Let us define a function $\tau(x, P; \mathcal{F})$ as the determinant of $(d+1) \times (d+1)$-dimensional matrix

$$\tau(x, P; \mathcal{F}) = \det \begin{pmatrix} \hat{\theta}(\hat{A}(P) + \hat{U}x + \hat{Z}_0) & \cdots & \hat{\theta}(\hat{A}(P) + \hat{U}x + \hat{Z}_d) \\ M_{1,0} & \cdots & M_{1,d} \\ \cdot & \cdot & \cdot \\ M_{n,0} & \cdots & M_{d,d} \end{pmatrix}. \tag{4.14}$$

Then, the common eigenfunction of the commuting operators can be represented in the form

$$\hat{\psi}_{BA} = \frac{\tau(x, P\,; \mathcal{F})}{\tau(x, P_+; \mathcal{F})} e^{x\Omega(P)} \tag{4.15}$$

The rest of the arguments proving the lemma are the same, as in the smooth case.

Lemma 4.3 *There exist g-dimensional vectors $V_m = \{V_{m,k}\}$ and constants v_m such that the abelian functions $F_m = \mathrm{res}_T \mathcal{L}^m$ are equal to*

$$F_m(Z) = v_m + \Delta_U \left(\partial_{V_m} \ln \theta(Z)\right), \tag{4.16}$$

where $\partial_{V_m} = \sum_{k=1}^{g} V_{m,k} \partial_{z_k}$.

Proof. The proof is identical to that of Lemma 3.6 in [12]. Recall that the functions F_n are abelian functions with simple poles at the divisors Θ and Θ_U. In order to prove the statement of the lemma it is enough to show that $F_n = \Delta_U Q_n$, where Q_n is a meromorphic function with a pole along Θ. Indeed, if Q_n exists, then, for any vector λ in the period lattice, we have $Q_n(Z + \lambda) = Q_n(Z) + c_{n,\lambda}$. There is no abelian function with a simple

pole on Θ. Hence, there exists a constant q_n and two g-dimensional vectors l_n, V_n, such that $Q_n = q_n + (l_n, Z) + (V_n, h(Z))$, where $h(Z)$ is a vector with the coordinates $\partial_{z_i} \ln \theta$. Therefore, $F_n = (l_n, U) + (V_n, \Delta_U)h$.

Let $\psi(x, Z, k)$ be the formal Baker-Akhiezer function defined in the previous lemma. Then the coefficients $\varphi_s(Z)$ of the corresponding wave operator Φ are global meromorphic functions with poles on Θ.

The left and right action of pseudo-difference operators are formally adjoint, i.e., for any two operators the equality $(k^{-x}\mathcal{D}_1)(\mathcal{D}_2 k^x) = k^{-x}(\mathcal{D}_1\mathcal{D}_2 k^x) + (T-1)(k^{-x}(\mathcal{D}_3 k^x))$ holds. Here $\mathcal{D}_3$ is a pseudo-difference operator whose coefficients are difference polynomials in the coefficients of $\mathcal{D}_1$ and $\mathcal{D}_2$. Therefore, from (3.9-3.13) it follows that

$$\psi^+\psi = 1 + \sum_{s=2}^{\infty} F_{s-1}k^{-s} = 1 + \Delta\left(\sum_{s=2}^{\infty} Q_s k^{-s}\right). \tag{4.17}$$

The coefficients of the series Q are difference polynomials in the coefficients φ_s of the wave operator. Therefore, they are global meromorphic functions of Z with poles on Θ. Lemma is proved.

In order to complete the proof of our main result we need one more standard fact of the $2D$ Toda lattice theory: flows of the $2D$ Toda lattice hierarchy define deformations of the commutative rings $\mathcal{A}$ of ordinary linear difference operators. The spectral curve is invariant under these flows. There are two sets of $2D$ Toda hierarchy flows. Each of them is isomorphic to the KP hierarchy. For a given spectral curve Γ the orbits of the KP hierarchy are isomorphic to the generalized Jacobian $J(\Gamma) = \mathrm{Pic}^0(\Gamma)$, which is the equivalence classes of zero degree divisors on the spectral curve. (see [3, 18, 20, 26]).

The part of $2D$ hierarchy we are going to use is a system of commuting differential equation for a pseudo-difference operator $\mathcal{L}$

$$\partial_{t_n}\mathcal{L} = [\mathcal{L}^n_+ + F_n, \mathcal{L}] = -[\mathcal{L}^n_- - F_n, \mathcal{L}]\,. \tag{4.18}$$

The coefficient w_0 of $\mathcal{L}$ in (3.1) equals

$$w_0 = -\Delta_U \xi_1 = -u. \tag{4.19}$$

Therefore, (4.18) and equations (3.5, 4.16) imply

$$\partial_{t_n} u = -\Delta_U F^1_n = -\partial_V F_n = -\partial_V [\Delta_U \partial_{V_n} \ln \theta(Z)\,, \tag{4.20}$$

where V_n is the vector defined in (4.16).

Equations (4.20) identify the tangent vector ∂_{t_n} to the orbit of the KP hierarchy with the tangent vector ∂_{V_n} to the abelian variety X. Hence, for a generic $Z \notin \Sigma$, the orbit of the KP flows of the ring $\mathcal{A}^Z$ is in X, i.e. it defines an holomorphic imbedding:

$$i_Z : J(\Gamma) \longmapsto X. \tag{4.21}$$

From (4.21) it follows that $J(\Gamma)$ is *compact*.

The generalized Jacobian of an algebraic curve is compact if and only if the curve is *smooth* ([27]). On a smooth algebraic curve a torsion-free rank 1 sheaf is a line bundle, i.e. $\overline{\mathrm{Pic}}(\Gamma) = J(\Gamma)$. Then (4.1) and the dimension arguments imply that i_Z is an isomorphism and the map j is inverse to i_Z. Theorem 1.1 is proved.

5 Fully discrete case

In this section we present the proof of Theorem 1.2. As above, we begin with the proof of the implication $(A) \longmapsto (C)$. We would like to mention that equation (1.18) can be derived as a necessary condition for the existence of a solution of (1.14), which is meromorphic in any of the variables m, n or in their linear combinations. For further use, let us introduce the variables

$$x = m - n, \quad \nu = m + n - 1\,. \tag{5.1}$$

In these variables equation (1.14) takes the form

$$\psi(x-1,\nu) = \psi(x+1,\nu) + u(x,\nu)\psi(x,\nu-1)\,. \tag{5.2}$$

Let $\tau(x,\nu)$ be a holomorphic function of the variable x in some *translational invariant* domain $\mathcal{D} = T\mathcal{D} \in \mathbb{C}$, where $T : x \to x+1$. Suppose that for each ν the function τ in $\mathcal{D}$ has a simple root $\eta(\nu)$ such that

$$\tau(\eta(\nu)+1,\nu-1)\,\tau(\eta(\nu)-1,\nu-1) \neq 0. \tag{5.3}$$

Lemma 5.1 *If equation (5.2) with the potential*

$$u = \frac{\tau(x,\nu+1)\,\tau(x,\nu-1)}{\tau(x-1,\nu)\,\tau(x+1,\nu)} \tag{5.4}$$

has a meromorphic in $\mathcal{D}$ solution $\psi(x,\nu)$ such that it has a simple pole at $\eta(\nu)$, and regular at $\eta(\nu+1)-1, \eta(\nu+1)+1$, then the equation

$$\frac{\tau(\eta+1,\nu+1)\,\tau(\eta-2,\nu)\,\tau(\eta+1,\nu-1)}{\tau(\eta-1,\nu+1)\,\tau(\eta+2,\nu)\,\tau(\eta-1,\nu-1)} = -1\,, \quad \eta = \eta(\nu) \tag{5.5}$$

holds.

Proof. The substitution in (5.2) of the Laurent expansion (2.3) for ψ (with coefficients depending on ν), and the expansion

$$\tau(x,\nu) = v_0(\nu)\,(x-\eta(\nu)) + O((x-\eta(\nu))^2)\,, \tag{5.6}$$

gives the following system of equations.

From the vanishing of the residues at $\eta+1$ and $\eta-1$ of the left hand side of (5.2) we get

$$\alpha(\nu) \;=\; \frac{\tau(\eta+1,\nu+1)\,\tau(\eta+1,\nu-1)}{\tau(\eta+2,\nu)\,v_0(\nu)}\,\psi(\eta+1,\nu-1), \tag{5.7}$$

$$-\alpha(\nu) \;=\; \frac{\tau(\eta-1,\nu+1)\,\tau(\eta-1,\nu-1)}{\tau(\eta-2,\nu)\,v_0(\nu)}\,\psi(\eta-1,\nu-1), \tag{5.8}$$

The evaluation of (5.2) at $x = \eta(\nu+1)$ gives the equation

$$\psi(\eta(\nu+1)-1,\nu) = \psi(\eta(\nu+1)+1,\nu). \tag{5.9}$$

Equations (5.7,5.8,5.9) directly imply (5.5). The Lemma is proved.

Equation (5.5) for τ of the form $\theta(mU+nV+Z)$ coincides with (1.18), i.e., the implication $(A) \longmapsto (C)$ is proved.

The next step is to show that equations (5.5) are sufficient for local existence of wave solutions with coefficients having poles only at zeros of τ. The wave solutions of (5.2) are solutions of the form

$$\psi(x,\nu,k) = k^{\nu}\left(1+\sum_{s=1}^{\infty}\xi_s(x,\nu)\,k^{-s}\right). \tag{5.10}$$

Lemma 5.2 *Suppose that $\tau(x,\nu)$ is holomorphic in a domain $\mathcal{D}$ of the form (2.7) where it has simple zeros, for which condition (5.3) and equation (5.5) hold. Then there exist meromorphic wave solutions of equation (5.2) that have simple poles at zeros of τ and are holomorphic everywhere else.*

Proof. Substitution of (5.10) into (5.2) gives a recurrent system of equations

$$\xi_{s+1}(x-1,\nu) - \xi_{s+1}(x+1,\nu) = u(x,\nu)\,\xi_s(x,\nu-1). \tag{5.11}$$

Under the assumption that $\mathcal{D}$ is a disconnected union of small disks, ξ_{s+1} can be defined as an arbitrary meromorphic function in D_0 and then extended on $\mathcal{D}$ with the help of (5.11). Our goal is to prove by induction that (5.11) has meromorphic solutions with simple poles only at the zeros of τ.

Suppose that $\xi_s(x,\nu)$ has a simple pole at $x=\eta(\nu)=\eta$

$$\xi_s = \frac{r_s}{x-\eta} + r_{s0} + \cdots. \tag{5.12}$$

The condition that $\xi_{s+1}(x,\nu)$ has no pole at $\eta+1$ is equivalent to the equation

$$r_{s+1}(\nu) = \frac{\tau(\eta+1,\nu+1)\,\tau(\eta+1,\nu-1)}{v_0\,\tau(\eta+2,\nu)}\,\xi_s(\eta+1,\nu-1). \tag{5.13}$$

The condition that $\xi_{s+1}(x,\nu)$ has no pole at $\eta-1$ is equivalent to the equation

$$-r_{s+1}(\nu) = \frac{\tau(\eta-1,\nu+1)\,\tau(\eta-1,\nu-1)}{v_0\,\tau(\eta-2,\nu)}\,\xi_s(\eta-1,\nu-1). \tag{5.14}$$

Using (5.11) for $s-1$, and the equation $u(\eta,\nu-1)=0$, we get

$$\xi_s(\eta-1,\nu-1) = \xi_s(\eta+1,\nu-1). \tag{5.15}$$

Then, equation (5.5) imply that two different expressions for $r_{s+1}(\nu)$ obtained from (5.13) and (5.14) do in fact coincide. The lemma is proved.

Normalization problem. As before, our goal is to show that wave solutions can be defined uniquely up to a constant factor with the help of certain quasi-periodicity conditions. That

requires the global existence of the wave functions along certain affine subspaces. In what follows we use affine subspaces in the direction of the vector $2W = U - V$. The *singular locus* Σ which controls the obstruction for the global existence of the wave solution on X is the maximal T_{U-V}-invariant subset of the theta-divisor, and which is not T_U or T_V-invaiant

$$\Sigma = \left\{ Z \;\Big|\; \frac{\theta(k(U-V)+Z)}{\theta(Z+U)} = 0, \quad \frac{k(U-V)+Z)}{\theta(Z+V)} = 0 \quad \forall\, k \in \mathbb{Z} \right\}. \tag{5.16}$$

Surprisingly it turns out that in the fully discrete case the proof of the statement that the singular locus is in fact empty can be obtained at much earlier stage than in the continuous or semi-continuous cases.

Let $Y = \langle (U-V)\, k \rangle$ be the Zariski closure of the group $\{(U-V)\, k \mid k \in \mathbb{Z}\}$ in X. It is generated by its irreducible component Y^0, containing 0, and by the point W_0 of finite order in X, such that $2W - W_0 \in Y^0$, $NW_0 = \lambda_0 \in \Lambda$. Shifting Z if needed, we may assume, without loss of generality, that 0 is not in the singular locus. Then $Y \cap \Sigma = \emptyset$.

Let $\tau(z,\nu)$ be a function defined by the formula

$$\tau(z,\nu) = \theta\left(z + \frac{\nu}{2}(U+V)\right), \quad z \in \mathcal{C}. \tag{5.17}$$

Here and below $\mathcal{C}$ is a union of affine subspaces, that are preimages of irreducible components of Y under the projection $\pi : \mathbb{C}^g \to X = \mathbb{C}^g/\Lambda$.

The restriction of equation (1.18) onto Y gives the equation

$$\frac{\tau(z+W,\nu+1)\,\tau(z-2W,\nu)\,\tau(z+W,\nu-1)}{\tau(z-W,\nu+1)\,\tau(z+2W,\nu)\,\tau(z-W,\nu-1)} = -1\,, \tag{5.18}$$

which is valid on the divisor $\mathcal{T}^\nu = \{\, z \in \mathcal{C} \mid \tau(z,\nu) = 0\}$.

The function

$$u = \frac{\tau(z,\nu+1)\,\tau(z,\nu-1)}{\tau(z-W,\nu)\,\tau(z+W,\nu)} \tag{5.19}$$

is periodic with respect to the lattice $\Lambda_W = \Lambda \cap \mathcal{C}$. The latter is generated by the sublattice $\Lambda_W^0 = \Lambda \cap \mathbb{C}^d$, where $\mathbb{C}^d$ is a linear subspace in $\mathbb{C}^g$, that is preimage of Y_W^0, and the vector $\lambda_0 = NW_0 \in \Lambda$. For fixed ν, the function $u(z,t)$ has simple poles on the divisors $\mathcal{T}^\nu \pm W$.

Lemma 5.3 *Let $\tau(z,\nu)$ be a sequence of non-trivial holomorphic functions on $\mathcal{C}$ such that $u(z,\nu)$ given by (5.19) is periodic with respect to Λ_W. Suppose that equation (5.18) holds. Then there exist wave solutions $\psi(z,\nu,k) = k^\nu \phi(z,\nu,k)$ of the equation*

$$\psi(z-W,\nu,k) = \psi(z+W,\nu,k) + u(z,\nu)\psi(z,\nu-1,k)\,, \tag{5.20}$$

such that:

(i) the coefficients $\xi_s(z,\nu)$ of the formal series

$$\phi(z,\nu,k) = 1 + \sum_{s=1}^{\infty} \xi_s(z,\nu)\, k^{-s}, \tag{5.21}$$

are meromorphic functions of the variable $z \in \mathcal{C}$ with simple poles at the divisor $\mathcal{T}^\nu$, i.e.

$$\xi_s(z,\nu) = \frac{\tau_s(z,\nu)}{\tau(z,\nu)}, \tag{5.22}$$

where $\tau_s(z,\nu)$ is now a holomorphic function;

(ii) $\xi_s(z,\nu)$ satisfy the following monodromy properties

$$\xi_s(z+\lambda,\nu) - \xi_s(z,\nu) = \sum_{i=1}^{s} B^{\lambda}_{i,\,\nu-s+i}\, \xi_{s-i}(z,\nu)\,, \quad \lambda \in \Lambda_W, \tag{5.23}$$

where $B^{\lambda}_{i,\nu}$ are z-independent.

Proof. The functions $\xi_s(z,\nu)$ are defined recursively by the equations

$$\xi_{s+1}(z-W,\nu) - \xi_{s+1}(z+W,\nu) = u(z,\nu)\, \xi_s(z,\nu-1). \tag{5.24}$$

We will now prove lemma by induction in s. Let us assume inductively that that for $r \le s$ the functions ξ_r are known and satisfy (5.23). Then, we define the residue of ξ_{s+1} on $\mathcal{T}^\nu$ by formulae

$$\tau^0_{s+1}(z,\nu) \;=\; \frac{\tau(z+W,\nu+1)\,\tau_s(z+W,\nu-1)}{\tau(z+2W,\nu)}, \quad z \in \mathcal{T}^\nu, \tag{5.25}$$

$$-\tau^0_{s+1}(z,\nu) \;=\; \frac{\tau(z-W,\nu+1)\,\tau_s(z-W,\nu-1)}{\tau(z-2W,\nu)}, \quad z \in \mathcal{T}^\nu\,, \tag{5.26}$$

which, as follows from (5.13) and (5.14), coincide. The expression (5.25) is certainly holomorphic when $\tau(z+2W)$ is non-zero, i.e. is holomorphic outside of $\mathcal{T}^\nu \cap (\mathcal{T}^\nu - 2W)$. Similarly from (5.26) we see that $\tau^0_{s+1}(z,\nu)$ is holomorphic away from $\mathcal{T}^\nu \cap (\mathcal{T}^\nu + 2W)$.

We claim that $\tau^0_{s+1}(z,\nu)$ is holomorphic everywhere on $\mathcal{T}^\nu$. Indeed, by definition of Y, the closure of the abelian subgroup generated by $2W$ is everywhere dense. Thus for any $z \in \mathcal{T}^\nu$ there must exist some $N \in \mathbb{N}$ such that $z - 2(N+1)W \notin \mathcal{T}^\nu$; let N moreover be the minimal such N. From (5.26) it then follows that $\tau^0_{s+1}(z,\nu)$ can be extended holomorphically to the point $z-2NW$. Thus expression (5.25) must also be holomorphic at $z-2NW$; since its denominator there vanishes, it means that the numerator must also vanish. But this expression is equal to the numerator of (5.26) at $z-2(N-1)W$; thus τ^0_{s+1} defined from (5.26) is also holomorphic at $z-2(N-1)W$ (the numerator vanishes, and the vanishing order of the denominator is one, since we are talking exactly about points on its vanishing divisor). Note that we did not quite need the fact $z-2(N+1)W \notin \mathcal{T}^\nu$ itself, but the consequences of the minimality of N, i.e., $z-2kW \in \mathcal{T}^\nu$, $0 \le k \le N$, and the holomorphicity of $\tau^0_{s+1}(z,\nu)$ at $z-2NW$." Therefore, in the same way, by replacing N by $N-1$, we can then deduce holomorphicity $\tau^0_{s+1}(z,\nu)$ at $z-2(N-2)W$ and, repeating the process N times, at z.

Recall once again that that an analytic function on an analytic divisor in $\mathbb{C}^d$ has a holomorphic extension to all of $\mathbb{C}^d$ ([24]). Therefore, there exists a holomorphic function

$\widetilde{\tau}_{s+1}(z,\nu)$ extending the $\tau^0_{s+1}(z,\nu)$. Consider then the function $\chi_{s+1}(z,\nu) = \widetilde{\tau}_{s+1}(z,\nu)/\tau(z,\nu)$, holomorphic outside of $\mathcal{T}^\nu$.

From (5.23) and (5.25) it follows that the function

$$f^\lambda_{s+1}(z,\nu) = \chi_{s+1}(z+\lambda,\nu) - \chi_{s+1}(z,\nu) - \sum_{i=1}^{s} B^\lambda_{i,\,\nu-1-s+i}\,\xi_{s+1-i}(z,\nu) \tag{5.27}$$

vanishes at the divisor $\mathcal{T}^\nu$. Hence, it is a holomorphic function. It satisfies the twisted homomorphism relations

$$f^{\lambda+\mu}_{s+1}(z,\nu) = f^\lambda_{s+1}(z+\mu,\nu) + f^\mu_{s+1}(z,\nu), \tag{5.28}$$

i.e., it defines an element of the first cohomology group of Λ_0 with coefficients in the sheaf of holomorphic functions, $f \in H^1_{gr}(\Lambda_0, H^0(\mathbb{C}^d,\mathcal{O}))$. Once again using the same arguments, as that used in the proof of the part (b) of the Lemma 12 in [3], we get that there exists a holomorphic function $h_{s+1}(z,\nu)$ such that

$$f^\lambda_{s+1}(z,\nu) = h_{s+1}(z+\lambda,\nu) - h_{s+1}(z,\nu) + \widetilde{B}^\lambda_{s+1,\,\nu}, \tag{5.29}$$

where $\widetilde{B}^{\lambda,}_{s+1,\,\nu}$ is z-independent. Hence, the function $\zeta_{s+1} = \chi_{s+1} + h_{s+1}$ has the following monodromy properties

$$\zeta_{s+1}(z+\lambda,\nu) - \zeta_{s+1}(z,\nu) = \widetilde{B}^\lambda_{s+1,\nu} + \sum_{i=1}^{s} B^\lambda_{i,\,\nu-1-s+i}\,\xi_{s+1-i}(z,\nu)\,. \tag{5.30}$$

Let us consider the function R_{s+1} defined by the equation

$$R_{s+1} = \zeta_{s+1}(z-W,\nu) - \zeta_{s+1}(z+W,\nu) - u(z,\nu)\,\xi_s(z,\nu-1)\,. \tag{5.31}$$

Equation (5.25) and (5.26) imply that the r.h.s of (5.31) has no pole at $\mathcal{T}^\nu \pm W$. Hence, $R_{s+1}(z,\nu)$ is a holomorphic function of z. From (5.23,5.30) it follows that it is periodic with respect to the lattice Λ_W, i.e., it is a function on Y. Therefore, R_{s+1} is a constant (z-independent) on each of the connected components of $\mathcal{C}$.

Hence, the function

$$\xi_{s+1}(z,\nu) = \zeta_{s+1}(z,\nu) + l_{s+1}(z,\nu) + c_{s+1}(\nu)\,, \tag{5.32}$$

where $c_{s+1}(\nu)$ is a constant, and l_{s+1} is a linear form such that

$$l_{s+1}(2W,\nu) = -R_{s+1}(\nu)\,,$$

is a solution of (5.24). It satisfies the monodromy relations (5.23) with

$$B^\lambda_{s+1,\,\nu} = \widetilde{B}^\lambda_{s+1,\,\nu} + l_{s+1}(\lambda,\nu)\,. \tag{5.33}$$

The induction step is completed and thus the lemma is proven.

On each step the ambiguity in the construction of ξ_{s+1} is a choice of linear form $l_{s+1}(z,\nu)$ and constants $c_{s+1}(\nu)$. As in Section 2, we would like to fix this ambiguity by normalizing monodromy coefficients $B_{i,\nu}^{\lambda}$ for a set of linear independent vectors $\lambda_0,\lambda_1,\ldots,\lambda_d\in\Lambda_W$. It turns out that in the fully discrete case there is an obstruction for that. This obstruction is a possibility of the existence of periodic solutions of (5.24), $\xi_{s+1}(z+\lambda,\nu)=\xi_{s+1}(z,\nu),\ \lambda\in\Lambda_W$, for $0\le s\le r-1$.

Note, that there are no periodic solutions of (5.24) for all s. Indeed, the functions $\xi_s(z,\nu)$ as solutions of non-homogeneous equations are linear independent. Suppose not. Take a smallest nontrivial linear relation among $\xi_s(z,\nu)$, and apply (5.24) to obtain a smaller linear relation. The space of meromorphic functions on Y with simple pole at $\mathcal{T}^\nu$ is finite-dimensional. Hence, there exists minimal r such that equation (5.24) for $s=r$ has no periodic solutions.

Lemma 5.4 *Let $\lambda_0,\lambda_1,\ldots,\lambda_d$ be a set of linear independent vectors in Λ_W. Suppose equations (5.24) has periodic solutions for $s<r$ and has a quasi-periodic solution ξ_r whose monodromy relations for λ_j have the form*

$$\xi_r(z+\lambda_j,\nu)-\xi_r(z,\nu)=b^{\lambda_j},\quad j=0,\ldots,d, \tag{5.34}$$

where b^{λ_i} are constants such that there is no linear form $l(z)$ on Y with $l(\lambda_j)=b^{\lambda_j}$ and $l(2W)=0$. Then for all s equations (5.24) has solutions of the form (5.22) satisfying (5.23) with $B_{i,\nu}^{\lambda_j}=b^{\lambda_j}\delta_{i,r}$, i.e.,

$$\xi_s(z+\lambda_j,\nu)-\xi_s(z,\nu)=b^{\lambda_j}\xi_{s-r}(z,\nu). \tag{5.35}$$

Proof. We will now prove the lemma by induction in $s\ge r$. Let us assume inductively that ξ_{s-r} is known, and for $1\le i\le r$ there are solutions $\tilde\xi_{s-r+i}$ of (5.24) satisfying (5.23) with $B_{i,\nu}^{\lambda_j}=b^{\lambda_j}\delta_{i,r}$. Then, according to the previous lemma, there exists a solution $\tilde\xi_{s+1}$ of (5.24) having the form (5.22) and satisfying monodromy relations (5.23), which for λ_j have the form

$$\tilde\xi_{s+1}(z+\lambda_j,\nu)-\tilde\xi_{s+1}(z,\nu)=b^{\lambda_j}\tilde\xi_{s-r+1}(z,\nu)+B_{s+1,\nu}^{\lambda_j}\,. \tag{5.36}$$

If ξ_{s-r} is fixed, then the general quasi-periodic solution ξ_{s-r+1} with the normalized monodromy relations is of the form

$$\xi_{s-r+1}(z,\nu)=\widetilde{\xi}_{s-r+1}(z,\nu)+c_{s-r+1}(\nu)\,, \tag{5.37}$$

where c_{s-r+1} are constants on each component of $\mathcal{C}$. It is easy to see that under the transformation (5.37) the functions $\widetilde{\xi}_{s-r+i}$ get transformed to

$$\xi_{s-r+i}(z,\nu)=\widetilde{\xi}_{s-r+i}(z,\nu)+c_{s-r+1}(\nu-i+1)\,\xi_{i-1}(z,\nu)\,. \tag{5.38}$$

This transformation does not change the monodromy properties of ξ_{s-r+i} for $i\le r$, but changes the monodromy property of ξ_{s+1}:

$$\xi_{s+1}(z+\lambda_j,\nu)-\xi_{s+1}(z,\nu)=b^{\lambda_j}\xi_{s-r+1}(z,\nu)+b^{\lambda_j}(c_{s-r+1}(\nu-r)-c_{s-r+1}(\nu))+B_{s+1,\nu}^{\lambda_j} \tag{5.39}$$

Recall, that $\widetilde{\xi}_{s+1}$ was defined up to a linear form $l_{s+1}(z,\nu)$ which vanishes on $2W$. Therefore the normalization of the monodromy relations for ξ_{s+1} uniquely defines this form and the differences $(c_{s-r+1}(\nu-r)-c_{s-r+1}(\nu))$. The induction step is completed and the lemma is thus proven.

Note, the following important fact: if ξ_{s-r} is fixed then ξ_{s-r+1}, such that there exists quasi-periodic solution ξ_{s+1} with normalized monodromy properties, is defined uniquely up to the transformation:

$$\xi_{s-r+1}(z,\nu)\longmapsto \xi_{s-r+1}(z,\nu)+c_{s-r+1}(\nu),\quad c_{s-r+1}(\nu+r)=c_{s-r+1}(\nu). \tag{5.40}$$

Our next goal is to show that the assumption of Lemma 5.4 holds for some r, and then to prove that the singular locus Σ is in fact empty.

Shifting $z\to Z+z$, we get, as a direct corollary of Lemma 5.3, that: if $Z\notin \cup_{i=0}^{s-1}(\Sigma_0-iV)$, where $\Sigma_0=\bigcap_{k\in\mathbb{Z}}T^k_{U-V}\Theta$, then there exist holomorphic functions $\tau_s(Z+z)$, which are *local* functions of the variable $Z\in\mathbb{C}^g$ and *global* function of the variable $z\in\mathcal{C}$, such that the equations

$$\frac{\tau_s(Z)}{\theta(Z)}-\frac{\tau_s(Z+U-V)}{\theta(Z+U-V)}=\frac{\theta(Z+U)\,\tau_{s-1}(Z-V)}{\theta(Z)\,\theta(Z+U-V)}, \tag{5.41}$$

holds, and the functions $\xi_s=\tau_s/\theta$ satisfy the monodromy relations

$$\xi_s(Z+z+\lambda)-\xi_s(Z+z)=\sum_{i=1}^{s}B_i^\lambda(Z)\,\xi_{s-i}(Z+z)\,,\quad \lambda\in\Lambda_W. \tag{5.42}$$

If ξ_{s-1} is fixed then ξ_s is unique up to the transformations

$$\xi_s(Z+z)\to\xi_s(Z+z)+l_s(Z,z)+c_s(Z), \tag{5.43}$$

where l_s is a linear form in z such that $l_s(Z,U-V)=0$, and $c_s(Z)$ are z-independent.

Let r be the minimal integer such that $\xi_1,\ldots,\xi_{r-1}$ are periodic functions of z with respect to Λ_W, and there is no periodic solution ξ_r of (5.41). As it was noted above, the functions τ_s are linear independent. Hence, $r\le h^0(Y,\theta|_Y)$.

If ξ_{r-1} is periodic, then the monodromy relation for ξ_r has the form

$$\xi_r(Z+z+\lambda)-\xi_r(Z+z)=B_r^\lambda(Z)\,,\quad \lambda\in\Lambda_W. \tag{5.44}$$

The function B_r^λ is independent of the ambiguities in the definition of $\xi_i,\ i<r$, and therefore, it is a well-defined holomorphic function of $Z\in X$ outside of the set $\cup_{i=1}^{r-1}(\Sigma-iV)$. The later is of codimension at least 2. Hence, by Hartogs' theorem $B_r^\lambda(Z)$ extends to a holomorphic function on X. Hence, it is a constant $B_r^\lambda(Z)=b^\lambda$. It was supposed that the function ξ_r can not be made periodic by transformation (5.43). Therefore, there is no linear form on $\mathcal{C}$ such that $l(\lambda)=b^\lambda,\ l(U-V)=0$, and the initial assumption of lemma 5.4 is proved.

Lemma 5.5 *If equation (1.18) is satisfied, then the singular locus $\Sigma\in\Theta$ is empty.*

Proof. The functions $\tau_1(Z+z)$ are defined as solutions of (5.41) along $\mathcal{C}$. The restriction of $\tau_1(Z)$ on Θ is given by the formulae (5.25, 5.26) for $s=0$, i.e.

$$\tau_1 = \frac{\theta(Z+U)\,\theta(Z-V)}{\theta(Z+U-V)} = -\frac{\theta(Z-U)\,\theta(Z+V)}{\theta(Z-U+V)} \tag{5.45}$$

Let us first show that Σ is invariant under the shift by rV (or equivalently by rU), where r is defined above (minimal integer such that there is no periodic solution ξ_r of (5.41). The functions $\tau_1(Z+z)$ are defined as solutions of (5.41) along $\mathcal{C}$, and a priori there are no relations between τ_1 defined for Z and its translates $Z-iV$. As shown in Lemma 5.4, the requirement that there exists ξ_{r+1} with normalized monodromy relations, defines τ_1 uniquely, up to the transformations (5.43) with $l_1=0$ and with rV-periodic c_1, i.e., $c_1(Z)=c_1(Z+rU)=c_1(Z+rV)$.

Let Z be in Σ and $Z+rV$ is not. Then τ_1 can be defined as a holomorphic function in the whole neighborhood of $(Z+rV)$. Therefore, $\tau_1(Z)$ can be defined as a single-valued holomorphic function of Z outside of Σ. Hence, by Hartogs' arguments it can be extended across Σ. The contradiction proves that $\Sigma=\Sigma+rV$.

By definition, Σ is not invariant under the shift by V. Hence, it is empty or $r>1$. Let $Z\in\Sigma$, then the r.h.s of equation (5.41) for $\tau_1(Z+V)$ vanishes. Therefore, ξ_1 is a constant along $\Sigma+V$. Using the transformation (5.43) we can make it to be equal zero on $\Sigma+V$, $\tau_1(Z+V)=0,\ Z\in\Sigma$. The same arguments applied consecutively show that we may assume that $\tau_i(Z+iV)=0,\ \ i\leq r-2$. For $i=r-1$, using in addition the equation $\theta(Z+rU)=0$ (which is due to the fact $\Sigma=\Sigma+rU$), we get that, up to the transformation (5.43), the function ξ_{r-1} has vanishing order on $\Sigma+(r-1)V$ such that the r.h.s of equation for ξ_r on $\Sigma+rV$ is zero. Hence, ξ_r can be defined as holomorphic function in the neighborhood of $\Sigma+rV$, and restricted on $\Sigma+rV$ is a constant. That contradicts to the assumption $b^\lambda\neq 0$, and thus the lemma is proven.

As shown above, if Σ is empty, then the functions τ_s can be defined as global holomorphic functions of $Z\in\mathbb{C}^g$. Then, as a corollary of the previous lemmas we get the following statement.

Lemma 5.6 *Let equation (1.18) for $\theta(Z)$ holds. Then there exists a formal solution*

$$\phi = 1+\sum_{s=1}^{\infty}\xi_s(Z)\,k^{-s} \tag{5.46}$$

of the equation

$$k\phi(Z+V,k) = k\phi(Z+U,k)+u(Z)\,\phi(Z,k)\,, \tag{5.47}$$

with

$$u(Z) = \frac{\theta(Z+U+V)\,\theta(Z)}{\theta(Z+U)\,\theta(Z+V)}, \tag{5.48}$$

such that:

(i) the coefficients ξ_s of the formal series ϕ are of the form $\xi_s=\tau_s/\theta$, where $\tau_s(Z)$ are holomorphic functions;

(ii) $\phi(Z,k)$ is quasi-periodic with respect to the lattice Λ and for the basis vectors λ_j in C^g its monodromy relations have the form

$$\phi(Z+\lambda_j) = (1 + b^{\lambda_j}\, k^{-1})\, \phi(Z,k), \quad j = 1, \ldots, g, \tag{5.49}$$

where b^{λ_j} are constants such that there is no linear form on $\mathbb{C}^g$ vanishing at λ_j and $U-V$, i.e., $l(\lambda_j) = \lambda(U-V) = 0$;

(iii) ϕ is unique up to the multiplication by a constant in Z factor.

Commuting difference operators. The formal series $\phi(Z,k)$ defines a unique pseudo-difference operator

$$\mathcal{L}(Z) = T + \sum_{s=0}^{\infty} w_s(Z)\, T^{-s}, \quad T = e^{\partial_m}, \tag{5.50}$$

such that the equation

$$\left(T + \sum_{s=0}^{N} w_s(Z + mU + nV)\, T^{-s} \right) \psi = k\psi\,. \tag{5.51}$$

holds. Here $\psi = k^{n+m}\phi(nV + mU + Z, k)$. The coefficients $w_s(Z)$ of $\mathcal{L}$ are meromorphic functions on the abelian variety X with poles along the divisors $T_U^{-i}\Theta = \Theta - iU,\ i \leq s+1$.

From equations $(5.47, 5.51)$ it follows that

$$\big((\Delta_1 \mathcal{L}^i)\, T_1 - (\Delta \mathcal{L}^i)\, T - [u, \mathcal{L}^i] \big)\, \psi = 0 \tag{5.52}$$

where $\Delta_1\mathcal{L}^i$ and $\Delta\mathcal{L}^i$ are pseudo-difference operator in T, whose coefficients are difference derivatives of the coefficients of $\mathcal{L}^i$ in the variables n and m respectively. Using the equation $(T_1 - T - u)\,\psi = 0$, we get

$$\big(\left(\Delta_1 \mathcal{L}^i\right) T - \left(\Delta \mathcal{L}^i\right) T + \left(\Delta_1 \mathcal{L}^i\right) u - [u, \mathcal{L}^i] \big)\, \psi = 0. \tag{5.53}$$

The operator in the left hand side of (5.53) is a pseudo-difference operator in the variable m. Therefore, it has to be equal to zero. Hence, we have the equation

$$\left(\Delta_0 \mathcal{L}^i\right) T + \left(\Delta_V \mathcal{L}^i\right) u - [u, \mathcal{L}^i] = 0, \ \ \Delta_0 = e^{\partial_V} - e^{\partial_U} \tag{5.54}$$

As before, the strictly positive difference part of the operator $\mathcal{L}^i$ we denote by $\mathcal{L}^i_+$. Then,

$$\left(\Delta_0 \mathcal{L}^i_+\right) T + \left(\Delta_V \mathcal{L}^i_+\right) u - [u, \mathcal{L}^i_+] = -\left(\Delta_0 \mathcal{L}^i_-\right) T - \left(\Delta_V \mathcal{L}^i_-\right) u + [u, \mathcal{L}^i_-] \tag{5.55}$$

The left hand side of (5.55) is a difference operator with non-vanishing coefficients only at the positive powers of T. The right hand side is a pseudo-difference operator of order 1. Therefore, it has the form $f_i T$. The coefficient f_i is easy expressed in terms of the leading coefficient $\mathcal{L}^i_-$. Finally we get the equation

$$\left(\Delta_0 \mathcal{L}^i_+\right) T + \left(\Delta_V \mathcal{L}^i_+\right) u - [u, \mathcal{L}^i_+] = -(\Delta_0 F_i)\, T, \tag{5.56}$$

where $F_i = \operatorname{res}\, \mathcal{L}^i$. The vanishing of the coefficient at T^0 in the right hand side of (5.55) implies the equation

$$\Delta_0 F_i^1 = -\left(\Delta_V F_i\right) u, \quad F_i^1 = \operatorname{res}\, \mathcal{L}^i T\,, \tag{5.57}$$

analogous to (3.5).

Lemma 5.7 *The abelian functions F_i have at most simple poles on the divisors Θ and Θ_U.*

The wave solution ψ define the unique operator Φ such that

$$\psi = \Phi k^{n+m}, \quad \Phi = 1 + \sum_{s=1}^{\infty} \varphi_s(Um + Vn + Z)\, T^{-s}. \tag{5.58}$$

The dual wave function

$$\psi^{+} = k^{-n-m} \left(1 + \sum_{s=1}^{\infty} \xi_s^{+}(Um + Vn + Z)\, k^{-s} \right) \tag{5.59}$$

is defined by the formula

$$\psi^{+} = k^{-n-m}\, T_1\, \Phi^{-1}\, T_1^{-1}. \tag{5.60}$$

It satisfies the equation

$$(T_1^{-1} - T^{-1} - u)\, \psi^{+} = 0, \tag{5.61}$$

which implies that the functions $\xi_s^{+}(Z)$ have the form $\xi_s^{+}(Z) = \tau_s^{+}(Z)/\theta(Z + U + V)$, where τ_s^{+} are holomorphic functions. Therefore, the functions $J_s(Z)$ such that

$$(\psi^{+} T_1)\, \psi = k + \sum_{s=1}^{\infty} J_s(Um + Vn + Z)\, k^{-s+1} \tag{5.62}$$

are meromorphic function on X with the simple poles at Θ and $T_U^{-1}\Theta = \Theta_U$.

From the definition of $\mathcal{L}$ it follows that

$$\mathrm{res}_k \left((\psi^{+} T_1)\, (\mathcal{L}^n \psi) \right) k^{-2} dk = \mathrm{res}_k \left((\psi^{+}\, T_1)\, \psi \right) k^{n-2} dk = J_n. \tag{5.63}$$

On the other hand, using the identity (3.12) we get

$$\mathrm{res}_k ((\psi^{+}\, T_1)\, \mathcal{L}^n \psi)\, k^{-2} dk = \mathrm{res}_k \left(k^{-n-m} \Phi^{-1} \right) \left(\mathcal{L}^n \Phi k^{n+m} \right) k^{-1} dk = \mathrm{res}_T \mathcal{L}^n = F_n. \tag{5.64}$$

Therefore, $F_n = J_n$ and the lemma is proved.

The rest of the proof of Theorem 1.2 is identical to that in the proof of Theorem 1.1. Namely: lemma 5.7 directly implies that for the generic $Z \in X$ linear combinations of operators $\mathcal{L}_{+}^{i}$ span commutative rings $\mathcal{A}^Z$ of ordinary difference operators. They define a spectral curve Γ with two smooth points $P_{\pm}$ and a map (4.1). The global existence of the wave function implies equations (4.16).

Equations (4.18) define the KP hierarchy deformations of these rings. From (4.18, 4.19), and equation (5.24) for $s = 0$ we get

$$\partial_{t_n} w_0 = \Delta_U F_n^1, \quad w_0 = -\Delta_U \xi_1, \;\; u = \Delta_0 \xi_1\,. \tag{5.65}$$

Then, (4.16) and (5.57) imply

$$\partial_{t_n} \ln u = \Delta_U \Delta_V (\partial_{V_n} \ln \theta). \tag{5.66}$$

By definition, u is given by the formula (1.15), i.e., $\ln u = \Delta_U \Delta_V \ln \theta$. Therefore, equation (5.66) identifies ∂_{t_n} with ∂_{V_n}. Hence, the orbit of the KP flows is in X. Hence the generalized Jacobian $J(\Gamma)$ of the spectral curve is compact and the spectral curve is smooth. As in the previous case these arguments complete the proof of the theorem.

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